Mode-Based Analysis II:
Variational, Geometric, and Categorical Structures

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Abstract

We extend the framework of mode-based analysis to variational calculus, differential geometry, and categorical structures. In this approach, convergence modes are treated as primary objects encoding operational procedures of approximation, rather than auxiliary constructions.

This shift leads to a reparameterization of fundamental analytical structures. Derivatives, admissible variations, metric notions, and integration acquire an implicit dependence on the class of admissible modes, while classical objects are recovered as invariants under sufficiently rich mode classes.

Beyond the structural formulation, we establish that mode-dependence leads to genuinely different analytical behavior. In particular, we show that: mode-dependent variational problems may admit minimizers that differ from classical ones, mode gradients may differ from classical gradients even for smooth functionals, and mode-dependent gradient flows may exhibit asymptotic behavior that is not captured by classical dynamics.

We further show that the mode differential can be interpreted as a restriction of the classical differential to admissible tangent cones, providing a geometric link between mode-based and classical structures.

In addition, we introduce minimal mode-dependent metric and measure structures sufficient to support variational and geometric constructions, and formulate a categorical organization of mode classes.

This yields a unified perspective in which analysis, geometry, and dynamics are governed by stability with respect to admissible operational procedures, rather than by fixed background constructions.

Keywords: convergence modes; mode-dependent calculus; operational convergence; stability of limits; mode-invariance; mode-sensitivity; variational principles; mode-dependent geometry; tangent cones; differential forms; curvature; metric structures; measure structures; categorical structures; solution clouds

Contents

1 Introduction

In classical analysis, variational calculus, differential geometry, and the theory of partial differential equations are built upon fixed notions of derivative, tangent space, and convergence. Integration and measure are likewise treated as background structures, assumed to be independent of the procedures by which limits are realized.

Within the framework of mode-based analysis [1], convergence is no longer treated as a procedure-independent concept. Instead, it is parameterized by *classes of admissible convergence modes*, which encode operational features such as discretization, ordering of increments, scaling relations, and regularization mechanisms.

As a consequence, the fundamental objects of analysis become mode-dependent. In particular:

- derivatives depend on operational procedures encoded by modes,
- admissible variations are restricted by mode constraints,
- metric structure arises from mode-dependent notions of distance and admissible paths,
- integration and measure acquire an implicit dependence on the mode class,
- geometric structures become anisotropic and mode-dependent.

This shift replaces fixed analytical structures by families of objects defined through stability with respect to admissible modes. Classical notions are recovered as invariants under sufficiently rich classes of modes.

The purpose of this paper is to extend the mode-based framework to three interconnected domains:

- variational principles, where admissible variations and Euler–Lagrange equations become mode-dependent,
- differential geometry, where tangent structures, differential forms, and curvature are defined via mode-stable differentials,
- categorical structures, where mode classes form a category and analytical constructions become functorial.

In addition, we introduce mode-dependent metric and measure structures in a minimal form, sufficient to support variational formulations and geometric constructions. A full development of mode-dependent measure theory and functional analysis is left for future work.

Beyond the structural formulation, the paper establishes several concrete results demonstrating that mode-dependence leads to genuinely different analytical behavior. In particular:

- mode-dependent total variation functionals may admit minimizers that are not minimizers of the classical functional,
- the mode gradient may differ from the classical gradient even for smooth functionals,
- mode-dependent gradient flows may exhibit asymptotic behavior that differs from classical gradient flows,
- the mode differential can be interpreted as a restriction of the classical differential to admissible tangent cones.

A fully explicit proof of non-equivalence of gradient flows in a standard Sobolev setting is provided in the Appendix.

This provides a unified structural perspective in which analysis, geometry, and dynamics are governed by stability with respect to classes of admissible operational procedures, rather than by fixed background constructions.

2 Scope of Part II

This part develops three structural layers of the mode-based framework:

- variational structure based on mode-admissible perturbations,
- geometric structure based on mode-dependent tangent cones,
- categorical structure organizing transformations of mode classes.

The goal is not to extend the framework to all domains, but to show that these three structures already emerge naturally from the notion of convergence modes and admit internally consistent formulations.

3 Mode-Dependent Variational Principles

3.1 Mode-Admissible Variations

Let $u : \mathbb{R}^n \to \mathbb{R}$.

Definition 1. *A variation δu is* mode-admissible *with respect to a class of modes R if it arises as:*

$$\delta u = \lim_{r \in R} \frac{u(r) - u}{\alpha_r},$$

for some scaling $\alpha_r \to 0$.

3.2 Well-posedness of Mode-Admissible Variations

The definition above requires a stability condition.

A variation is called *well-defined with respect to R* if the limit is stable under perturbations of the mode in the mode topology. That is, small changes in r do not affect the resulting variation.

This excludes purely formal variations that do not correspond to admissible operational procedures.

3.3 Mode-Dependent Functional

Definition 2. *A mode-dependent functional is:*

$$J_R[u] = \int_\Omega L(u, \nabla_R u)\, dx,$$

where $\nabla_R u$ is the mode-stable gradient.

3.4 Mode Euler–Lagrange Equation

We now derive the mode-dependent Euler–Lagrange equation.

Let
$$J_R[u] = \int_\Omega L(u, \nabla_R u)\, dx,$$
where L is sufficiently smooth and $\nabla_R u$ denotes the mode-stable gradient.

Let $u_\alpha = u + \alpha \delta u$, where δu is an R-admissible variation vanishing on $\partial\Omega$. Assume that differentiation with respect to α may be interchanged with mode-stable differentiation.

Then:
$$\frac{d}{d\alpha} J_R[u_\alpha]\Big|_{\alpha=0} = \int_\Omega \left(\partial_u L\, \delta u + \partial_{\nabla_R u} L \cdot \nabla_R(\delta u)\right)\, dx.$$

Using mode-dependent integration by parts,
$$\int_\Omega \partial_{\nabla_R u} L \cdot \nabla_R(\delta u)\, dx = -\int_\Omega \operatorname{div}_R(\partial_{\nabla_R u} L)\, \delta u\, dx,$$
because $\delta u|_{\partial\Omega} = 0$.

Therefore:
$$\frac{d}{d\alpha} J_R[u_\alpha]\Big|_{\alpha=0} = \int_\Omega (\partial_u L - \operatorname{div}_R(\partial_{\nabla_R u} L))\, \delta u\, dx.$$

Since this must vanish for all R-admissible variations δu, we obtain:
$$\partial_u L - \operatorname{div}_R\left(\partial_{\nabla_R u} L\right) = 0.$$

Theorem 1 (Mode Euler–Lagrange Equation)**.** *If u is stationary with respect to all R-admissible variations, then u satisfies*
$$\partial_u L - \operatorname{div}_R\left(\partial_{\nabla_R u} L\right) = 0.$$

Remark 1. *The classical Euler–Lagrange equation is recovered when R is a mode class for which $\nabla_R = \nabla$ and $\operatorname{div}_R = \operatorname{div}$.*

3.5 Mode-Dependent Gradient Flows

Definition 3. *A mode-dependent gradient flow is defined as:*
$$\partial_t u = -\nabla_R J_R[u],$$
where ∇_R is the mode-stable gradient.

Remark 2. *The evolution depends on the admissible class of modes and may differ for different operational structures.*

3.6 Minimizing Movements

Definition 4. *A discrete mode-gradient flow can be defined via minimizing movements:*
$$u_{k+1} \in \arg\min_v \left\{ J_R[v] + \frac{1}{2\tau} d_R^2(v, u_k) \right\}.$$

Theorem 2. *If the mode class R is sufficiently regular, the minimizing movement scheme converges to a mode-gradient flow.*

Remark 3. *Mode restrictions act as implicit regularization and may select different evolution trajectories.*

3.7 Example: Mode-Dependent Total Variation

Consider the functional:

$$J_R[u] = \int_\Omega |\nabla_R u|\, dx.$$

The corresponding Euler–Lagrange equation is:

$$-\operatorname{div}_R \left(\frac{\nabla_R u}{|\nabla_R u|} \right) = 0.$$

Different mode classes R lead to different admissible minimizers, reflecting different implicit regularization mechanisms.

This shows that even classical variational problems become mode-dependent when the differential structure is replaced by mode-stable differentials.

3.8 Remark on Integration and Measure

Remark 4. *In the present formulation, integrals are taken with respect to a fixed reference measure.*

A fully mode-dependent integration theory would require introducing a mode-measure μ_R, defined as a stable limit over admissible modes:

$$\mu_R(A) = \lim_{r \in R} \mu(A_r).$$

This extension is not developed here and is left for future work.

3.9 Non-Equivalence of Mode-Dependent Total Variation

The previous example shows formally that the total variation functional becomes mode-dependent when the classical gradient is replaced by the mode-stable gradient. We now show that this dependence is not merely notational: different mode classes may lead to genuinely different minimizing structures.

Definition 5 (Mode-stationary point)**.** *Let J_R be a mode-dependent functional. A function u is called R-stationary if*

$$\frac{d}{d\alpha} J_R[u + \alpha \delta u]\Big|_{\alpha=0} = 0$$

for all R-admissible variations δu.

Remark 5. *Under the assumptions of Theorem 1, R-stationarity is equivalent to satisfying the mode Euler–Lagrange equation.*

Let $\Omega \subset \mathbb{R}^2$ and consider the classical total variation functional

$$J[u] = \int_\Omega |\nabla u|\, dx.$$

Let R_{full} be a mode class for which all directions of approach are admissible, so that

$$\nabla_{R_{\text{full}}} u = \nabla u.$$

Let R_x be a suppression mode class in which only increments in the x-direction are admissible. Then the corresponding mode-gradient is

$$\nabla_{R_x} u = \partial_x^R u,$$

and the mode-dependent total variation becomes

$$J_{R_x}[u] = \int_\Omega |\partial_x^R u|\, dx.$$

The corresponding mode Euler–Lagrange equation is

$$-\partial_x^R \left(\frac{\partial_x^R u}{|\partial_x^R u|} \right) = 0,$$

whereas the classical total variation equation is

$$-\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0.$$

Theorem 3 (Non-equivalence of mode-dependent total variation)**.** *There exist functions u and mode classes R_1, R_2 such that u is minimizing for the mode-dependent total variation associated with R_1, but is not minimizing for the corresponding functional associated with R_2.*

Proof. We consider the functional J_R on the space $BV(\Omega)$ of functions of bounded variation, or on $L^1(\Omega)$ with finite total variation. Since $J_{R_x}[u] = 0$ and $J_{R_x} \geq 0$, u is a global minimizer in this class.

Let $\Omega = (0,1)^2$ and let

$$u(x, y) = y.$$

For the x-suppression mode class R_x, only increments in the x-direction are admissible. Hence

$$\partial_x^R u = 0.$$

Therefore

$$J_{R_x}[u] = \int_\Omega |\partial_x^R u|\, dx = 0.$$

Since J_{R_x} is nonnegative, u is a minimizer of J_{R_x} and hence is stationary with respect to all R_x-admissible variations.

For the full mode class R_{full}, however,

$$\nabla u = (0, 1),$$

and therefore

$$J_{R_{\text{full}}}[u] = \int_\Omega |\nabla u|\, dx = |\Omega| > 0.$$

Moreover, u is not a minimizer of the full total variation functional under boundary conditions that do not force variation in the y-direction. For example, among admissible functions with no prescribed boundary values, the constant functions have zero total variation, while $u(x, y) = y$ has positive total variation.

Thus u is minimizing for the R_x-dependent total variation, but not minimizing for the full total variation. Hence the mode-dependent variational problem is not equivalent to the classical one. □

Remark 6. *This example shows that the dependence on the mode class R is not a relabeling of the classical functional. The admissible differential structure changes the null directions of the functional and therefore changes the class of stationary and minimizing configurations.*

Remark 7. *In geometric terms, the mode tangent cone of R_x contains only the x-direction. Variations in the suppressed y-direction are invisible to the corresponding mode differential. As a result, functions that vary only in the suppressed direction may become stationary or minimizing for the mode-dependent functional, although they are nontrivial from the classical point of view.*

3.10 Regularized Mode-Dependent Total Variation

We assume that the mode-divergence operator div_R is defined as the adjoint of ∇_R with respect to the reference measure, consistent with the integration by parts formula introduced earlier.

To avoid singularities at points where the mode-gradient vanishes, one may consider the regularized mode-dependent total variation

$$J_{R,\varepsilon}[u] = \int_\Omega \sqrt{|\nabla_R u|^2 + \varepsilon^2}\, dx, \qquad \varepsilon > 0.$$

The corresponding Euler–Lagrange equation is

$$-\operatorname{div}_R\left(\frac{\nabla_R u}{\sqrt{|\nabla_R u|^2 + \varepsilon^2}}\right) = 0.$$

For the suppression class R_x, this becomes

$$-\partial_x^R\left(\frac{\partial_x^R u}{\sqrt{|\partial_x^R u|^2 + \varepsilon^2}}\right) = 0.$$

Let again $u(x,y) = y$. Since $\partial_x^R u = 0$, we obtain

$$-\partial_x^R\left(\frac{0}{\sqrt{\varepsilon^2}}\right) = 0.$$

Thus u is an exact stationary point of the regularized R_x-dependent total variation for every $\varepsilon > 0$.

For the full mode class, however,

$$\nabla u = (0,1),$$

and the functional detects the variation in the y-direction. Hence the regularized full total variation and the regularized R_x-dependent total variation have different minimizing structures.

Proposition 1 (Difference of null directions)**.** *For the suppression mode class R_x, any function independent of x satisfies*

$$\nabla_{R_x} u = 0,$$

and is therefore stationary for the regularized functional $J_{R_x,\varepsilon}$.

In contrast, the classical gradient detects variations in all directions, and the corresponding functional is sensitive to variations in y.

3.11 Mode-Dependent Gradient and Optimization

Definition 6 (Mode differential and gradient)**.** *Let $J_R[u]$ be a mode-dependent functional. The mode directional derivative is defined as*

$$D_R J_R[u](\delta u) = \lim_{\alpha \to 0} \frac{J_R[u + \alpha \delta u] - J_R[u]}{\alpha},$$

for all R-admissible variations δu, whenever the limit exists and is stable with respect to the mode topology.

If this functional is representable by an element of the admissible tangent structure, we denote this representative by $\nabla_R J_R[u]$ and call it the mode gradient.

Remark 8. *The gradient depends on the admissible class of modes and incorporates the operational structure of perturbations, including suppression, ordering, and regularization.*

Proposition 2. *If the functional J is classically differentiable and the mode class R is sufficiently rich, then:*

$$\nabla_R J = \nabla J.$$

Remark 9. *Different mode classes may induce different gradients, leading to distinct optimization dynamics even for the same functional.*

3.12 Mode-Gradient Descent

Definition 7. *A mode-dependent gradient descent scheme is defined as:*

$$u_{k+1} = u_k - \eta \nabla_R J_R[u_k].$$

Remark 10. *The convergence properties of the scheme depend on the mode class R, rather than solely on smoothness or convexity of J.*

3.13 Example: Non-Equivalence of Mode Gradient

We construct an explicit example where the mode gradient differs from the classical gradient.

Let $\Omega \subset \mathbb{R}^2$ and consider the functional

$$J[u] = \int_\Omega |\nabla u|^2 \, dx.$$

Then the classical gradient is

$$\nabla J[u] = -\Delta u.$$

Let R_x be a suppression mode class admitting only increments in the x-direction. Then the mode gradient depends only on variations along x, and we obtain

$$\nabla_{R_x} J[u] = -\partial_{xx} u.$$

Proposition 3 (Mode gradient differs from classical gradient)**.** *There exist functions u such that*

$$\nabla_{R_x} J[u] \neq \nabla J[u].$$

Proof. Let

$$u(x, y) = y^2.$$

Then

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0 + 2 = 2,$$

so

$$\nabla J[u] = -2.$$

However, since u does not depend on x,

$$\partial_{xx} u = 0,$$

and therefore

$$\nabla_{R_x} J[u] = 0.$$

Thus

$$\nabla_{R_x} J[u] \neq \nabla J[u].$$

□

Remark 11. *This example shows that the mode gradient captures only admissible directions of variation. Suppressed directions are invisible to the differential structure, leading to fundamentally different optimization dynamics.*

3.14 Relation to Classical Gradient via Tangent Cones

We now relate the mode gradient to the classical gradient via the geometry of the mode tangent cone.

Let $T_R(p)$ denote the mode tangent cone at a point p, defined by admissible directions of convergence.

Proposition 4 (Mode gradient as restricted differential)**.** *Let $J[u]$ be a functional that is classically Fréchet differentiable at u. Then the mode differential satisfies*

$$D_R J[u](\delta u) = DJ[u](\delta u)$$

for all R-admissible variations δu.

In particular, the mode gradient $\nabla_R J[u]$ is the restriction of the classical differential to the admissible tangent cone T_R.

Proof. By definition, R-admissible variations are limits of admissible increments. Since J is classically differentiable, its differential $DJ[u]$ exists and is continuous with respect to admissible perturbations.

Therefore, for any R-admissible variation δu, we have

$$D_R J[u](\delta u) = \lim_{\alpha \to 0} \frac{J[u + \alpha \delta u] - J[u]}{\alpha} = DJ[u](\delta u).$$

□

Remark 12. *The mode gradient does not introduce a new differential structure, but restricts the classical differential to admissible directions encoded by the mode class R.*

Remark 13. *If the admissible tangent cone T_R does not span the full tangent space, the mode gradient cannot detect variations outside T_R. This leads to degeneracies and non-equivalence with the classical gradient.*

Proposition 5 (Projection interpretation)**.** *Assume that the classical gradient $\nabla J[u]$ exists and that the tangent space admits an inner product structure.*

Then the mode gradient $\nabla_R J[u]$ can be interpreted as the projection of the classical gradient onto the admissible tangent cone T_R:

$$\nabla_R J[u] = \Pi_{T_R}(\nabla J[u]),$$

whenever such a projection is well-defined.

Remark 14. *In the case of suppression modes R_x, the tangent cone consists of directions parallel to the x-axis. The projection reduces to discarding all components of the classical gradient orthogonal to this direction.*

3.15 Example: Different Gradient Descent Dynamics

We now show that replacing the classical gradient by the mode gradient changes not only the differential object, but also the resulting optimization dynamics.

Let $\Omega \subset \mathbb{R}^2$ and consider

$$J[u] = \int_\Omega |\nabla u|^2 \, dx.$$

The classical gradient descent flow is

$$\partial_t u = \Delta u.$$

For the suppression mode class R_x, the mode gradient is

$$\nabla_{R_x} J[u] = -\partial_{xx} u,$$

and the corresponding mode-gradient flow is

$$\partial_t u = \partial_{xx} u.$$

Thus the classical flow is the two-dimensional heat equation, whereas the R_x-mode flow is a one-dimensional heat equation acting only in the x-direction.

Proposition 6 (Non-equivalence of gradient descent dynamics)**.** *There exist initial data for which the classical gradient flow converges to a different limit than the R_x-mode gradient flow.*

Proof. Let $\Omega = (0,1)^2$ with periodic boundary conditions, and take the initial condition

$$u_0(x, y) = \sin(2\pi y).$$

Under the classical gradient flow

$$\partial_t u = \Delta u,$$

we have

$$u(t, x, y) = e^{-4\pi^2 t} \sin(2\pi y),$$

so

$$u(t, \cdot, \cdot) \to 0$$

as $t \to \infty$.

Under the R_x-mode gradient flow

$$\partial_t u = \partial_{xx} u,$$

the same initial datum satisfies

$$\partial_{xx} u_0 = 0,$$

because u_0 is independent of x. Therefore the solution is stationary:

$$u(t, x, y) = \sin(2\pi y)$$

for all $t \geq 0$.

Hence the classical gradient flow converges to 0, while the R_x-mode gradient flow remains equal to the initial condition. The two optimization dynamics are therefore not equivalent. □

Remark 15. *This example shows that mode-dependent optimization may preserve components that are eliminated by the classical gradient flow. Suppression modes therefore act not merely as numerical restrictions, but as structural constraints on the space of admissible descent directions.*

Remark 16. *In computational terms, the mode class R determines which components of the state space are visible to descent. Consequently, convergence is governed by the operational structure encoded by R, rather than by the classical gradient alone.*

3.16 Mode-Stability and Convergence

Theorem 4 (Mode-Stability Convergence Principle)**.** *Let $J_R : X \to \mathbb{R} \cup \{+\infty\}$ be a mode-dependent functional defined on a space X (e.g., $L^2(\Omega)$ or $BV(\Omega)$), and consider the iterative scheme*

$$u_{k+1} = u_k - \eta \nabla_R J_R[u_k].$$

Assume:

- *the mode gradient $\nabla_R J_R$ exists on a domain $D \subset X$ and is stable with respect to the mode topology,*
- *the functional J_R is bounded from below,*
- *the step size $\eta > 0$ is sufficiently small,*
- *the sequence $\{u_k\}$ remains in a compact (or sequentially compact) subset of X with respect to the mode topology,*
- *the mode class R enforces consistent suppression and regularization patterns.*
- *the scheme satisfies a descent condition:*

$$J_R[u_{k+1}] \leq J_R[u_k] - c\|\nabla_R J_R[u_k]\|_R^2$$

for some constant $c > 0$,

Then the sequence $\{u_k\}$ admits a subsequence $\{u_{k_j}\}$ which converges in the mode topology induced by R to a limit u^* satisfying

$$\nabla_R J_R[u^*] = 0.$$

Sketch of proof. By the descent condition, the sequence $J_R[u_k]$ is non-increasing for sufficiently small η. The stability of $\nabla_R J_R$ ensures that the updates are consistent under perturbations of admissible modes.

Compactness (or sequential compactness) in the mode topology yields the existence of a convergent subsequence $u_{k_j} \to u^*$.

Passing to the limit in the iteration scheme using stability of the mode gradient implies

$$\nabla_R J_R[u^*] = 0.$$

□

Lemma 1 (Degeneracy of suppressed modes)**.** *Let R_x be a mode class admitting only increments in the x-direction. Then any function $u(x, y)$ independent of x satisfies*

$$\nabla_{R_x} u = 0.$$

Proof. All admissible increments Δu_r are computed using perturbations along the x-direction. If u does not depend on x, such increments vanish, and hence the mode-stable limit defining $\nabla_{R_x} u$ is zero. □

Remark 17. *The convergence mechanism is not determined solely by smoothness or convexity of J_R, but by the stability properties of the mode class R, which encode admissible perturbations and implicit regularization.*

Remark 18. *Different mode classes may induce different stationary points and different convergence behaviors for the same functional, even when the underlying functional form is identical.*

3.17 Relation to Minimizing Movements

Remark 19. *The mode-dependent gradient descent and minimizing movement scheme can be viewed as a generalization of the classical minimizing movements approach of De Giorgi.*

In the classical setting, evolution is defined via:

$$u_{k+1} \in \arg\min_{v} \left\{ J[v] + \frac{1}{2\tau} d^2(v, u_k) \right\},$$

with a fixed metric structure.

In the mode-based framework, both the functional J_R and the distance d_R depend on the class of modes, and the admissible variations are restricted accordingly.

Thus, minimizing movements become mode-dependent, and different mode classes may induce different evolution trajectories even for the same underlying functional.

Remark 20. *This provides a structural explanation for the dependence of gradient flows on discretization, regularization, and numerical schemes, which are encoded in the choice of mode class R.*

3.18 Relation to Γ-Convergence and Minimizing Movements

We relate the mode-based framework to the classical theory of Γ-convergence and minimizing movements.

Definition 8 (Mode family of functionals)**.** *Let R be a mode class. A family of functionals $\{J_r\}_{r\in R}$ is called a mode family if each J_r corresponds to a realization of the functional under the operational procedure encoded by r.*

Definition 9 (Mode limit functional)**.** *The mode-dependent functional J_R is said to be induced by the family $\{J_r\}_{r\in R}$ if*

$$J_R[u] = \lim_{r\in R} J_r[u],$$

whenever the limit exists and is stable with respect to the mode topology.

Proposition 7 (Relation to Γ-limits)**.** *Assume that the family $\{J_r\}_{r\in R}$ is equicoercive and admits a Γ-limit J with respect to a topology τ.*

If the mode topology is compatible with τ, then the mode functional J_R coincides with the Γ-limit:

$$J_R = J.$$

Sketch of proof. By definition, J_R is obtained as a stable limit over admissible modes. If the family $\{J_r\}$ is equicoercive and admits a Γ-limit J, then convergence of minimizers and stability of the functional follow from standard results of Γ-convergence.

Compatibility of the mode topology with τ ensures that the limiting procedure defining J_R selects the same limit as the Γ-limit. □

Remark 21. *This shows that classical Γ-convergence corresponds to a particular mode-invariant regime, where the class R encodes approximations compatible with the topology τ.*

Proposition 8 (Mode-dependent minimizing movements)**.** *Let $\{J_r\}_{r\in R}$ be a mode family of functionals and let d_R be a mode-dependent distance.*

Define the discrete scheme

$$u_{k+1} \in \arg\min_{v} \left\{ J_R[v] + \frac{1}{2\tau} d_R^2(v, u_k) \right\}.$$

Then the evolution depends on the mode class R, and different mode classes may lead to different limiting trajectories, even when the underlying family $\{J_r\}$ is the same.

Remark 22. *In the classical setting, minimizing movements are determined by a fixed metric and a fixed functional. In the mode-based framework, both the functional and the metric depend on R, providing a structural explanation for the dependence of evolution on discretization, regularization, and numerical schemes.*

Remark 23. *From this perspective, mode-dependence generalizes Γ-convergence by encoding not only the limiting functional, but also the admissible directions of variation and the induced geometry of the space.*

4 Mode-Dependent Metric Structure

4.1 Mode Distance

Definition 10. *Let R be a class of modes. A* mode-dependent distance *d_R is defined as:*

$$d_R(p, q) = \lim_{r \in R} \|p_r - q_r\|,$$

whenever the limit exists and is stable with respect to the topology on R.

Remark 24. *The distance depends on admissible increments and may be anisotropic or degenerate depending on the mode class.*

4.2 Mode Lipschitz Continuity

Definition 11. *A function f is* mode-Lipschitz *if there exists $C > 0$ such that:*

$$|f(x) - f(y)| \leq C\, d_R(x, y)$$

for all x, y.

4.3 Mode Length and Geodesics

Definition 12. *The length of a curve γ is:*

$$\ell_R(\gamma) = \lim_{r \in R} \sum_n \|\Delta x_n\|.$$

Definition 13. *A curve is* mode-geodesic *if it minimizes ℓ_R among admissible paths.*

Remark 25. *This induces a Finsler-type geometry determined by the mode class.*

5 Mode-Dependent Measure and Integration

5.1 Mode Measures

Definition 14. *Let R be a class of modes. A* mode-dependent measure *μ_R is defined as a stable limit:*

$$\mu_R(A) = \lim_{r \in R} \mu(A_r),$$

where A_r is the approximation of A induced by the mode r.

Remark 26. *The measure depends on the operational structure of approximation and may differ for different mode classes.*

5.2 Mode Integration

Definition 15. *The mode integral of a function f is defined as:*

$$\int f \, d\mu_R = \lim_{r \in R} \int f(r) \, d\mu,$$

whenever the limit exists and is stable.

Remark 27. *Integration depends on admissible discretizations and regularization patterns encoded in R.*

5.3 Compatibility with Classical Measure

Proposition 9. *If the mode class R is sufficiently rich and compatible with classical convergence, then:*

$$\mu_R = \mu, \qquad \int f \, d\mu_R = \int f \, d\mu.$$

Remark 28. *Classical measure theory is recovered as a mode-invariant case.*

5.4 Role in Variational Principles

Remark 29. *Mode-dependent measures provide the correct foundation for:*

- *variational functionals,*
- *weak formulations of PDE,*
- *flux integrals and conservation laws.*

A full development of mode-measure theory is beyond the scope of this work.

6 Mode-Dependent Geometry

6.1 Mode Tangent Cone

Definition 16. *The mode tangent cone at p is:*

$$T_p^R = \left\{ \lim_{n \to \infty} \frac{\Delta x_n}{\|\Delta x_n\|} : r \in R \right\}.$$

Remark 30. *For broad R, $T_p^R = S^{n-1}$. For suppression modes, T_p^R degenerates to coordinate directions.*

6.2 Structure of Mode Tangent Cones

The structure of T_p^R depends strongly on the mode class:

- For broad classes R, all directions are admissible: $T_p^R = S^{n-1}$.
- For suppression classes R_i, only coordinate directions survive: $T_p^R = \{\pm e_i\}$.
- For ordered modes, T_p^R reflects anisotropic hierarchies between directions.

Thus, the tangent structure is not fixed, but determined by the admissible convergence modes.

6.3 Mode Differential

Definition 17. *The mode differential is a linear map:*

$$d_R u(p) : T_p^R \to \mathbb{R}.$$

6.4 Interpretation of the Mode Differential

The mode differential can be interpreted as a linear functional on the tangent cone:

$$d_R u(p) : T_p^R \to \mathbb{R}.$$

In contrast to the classical differential, which acts on the full tangent space, the mode differential acts only on admissible directions.

Thus, the differential structure is determined by the geometry of admissible limits.

6.5 Mode Differential Forms

Definition 18. *A mode* 1*-form is:*

$$\omega_R = \sum_i a_i(x)\, dx_i^R,$$

where dx_i^R are mode-dependent differentials.

6.6 Mode Exterior Derivative

Definition 19.

$$d_R \omega_R = \sum_{i,j} \partial_i^R(a_j)\, dx_i^R \wedge dx_j^R.$$

6.7 Mode Curvature

Definition 20. *A mode connection:*

$$\nabla_R v = d_R v + \Gamma_R v.$$

Curvature:

$$\mathcal{R}_R = d_R \Gamma_R + \Gamma_R \wedge \Gamma_R.$$

Remark 31. *Curvature becomes mode-dependent and reflects anisotropy of convergence.*

7 Categorical Structure of Modes

Remark 32. *The categorical formulation presented in this section is intended as a structural organization of mode-dependent constructions. A fully rigorous categorical development is beyond the scope of this work and is left for future study.*

7.1 Category of Modes

Definition 21. *Define a category* **Mode***:*

- *Objects: mode classes R*
- *Morphisms: maps preserving suppression, ordering, and regularization*

7.2 Morphisms of Mode Classes

A morphism $f : R_1 \to R_2$ is required to preserve:

- suppression structure,
- ordering of axes,
- regularization patterns.

Such morphisms represent transformations of admissible operational procedures.

7.3 Functor of Differentials

Definition 22. *A functor:*

$$D : \mathbf{Mode} \to \mathbf{Vect}$$

maps:

- *$R \mapsto$ space of mode differentials*
- *morphisms $\mapsto$ pushforward maps*

7.4 Functor of Solution Clouds

Definition 23.

$$S : \mathbf{Mode} \to \mathbf{Set}, \quad R \mapsto S(R)$$

7.5 Mode-Invariance

Definition 24. *A PDE is mode-invariant if the functor S is constant on admissible mode classes, i.e.,*

$$S(R_1) \cong S(R_2)$$

for all admissible R_1, R_2.

Remark 33. *Mode-invariance corresponds to the existence of a natural isomorphism between the functor S evaluated on different mode classes.*

Mode-sensitivity corresponds to the failure of such naturality, providing a categorical interpretation of nonlinearity.

7.6 Mode-Sensitivity

Definition 25. *A PDE is mode-sensitive if:*

$$S(R_1) \ncong S(R_2).$$

Remark 34. *Nonlinearity can be interpreted as failure of categorical naturality.*

8 Conclusion

We have extended mode-based analysis beyond differential calculus to encompass variational principles, geometric structures, and categorical formulations.

Within this framework, fundamental analytical objects are no longer fixed, but arise as stable invariants with respect to classes of admissible convergence modes. In particular:

- variational principles depend on admissible operational perturbations,
- metric structure emerges from mode-dependent notions of distance and admissible paths,
- integration and measure acquire an implicit dependence on the mode class,
- geometric structures are determined by mode-dependent tangent cones and differentials,
- solution spaces and differentials organize functorially over mode classes.

In addition to this structural formulation, the results of the paper demonstrate that mode-dependence leads to genuinely different analytical behavior. In particular:

- mode-dependent total variation functionals may admit minimizers that are not minimizers of the classical functional,
- the mode gradient may differ from the classical gradient even for smooth functionals,
- mode-dependent gradient flows may exhibit asymptotic behavior that differs from classical gradient flows,
- the mode differential can be interpreted as a restriction of the classical differential to admissible tangent cones.

These results show that mode-dependence is not merely a reformulation of classical analysis, but introduces a structurally different framework in which admissible directions of convergence play a central role.

This leads to a unified perspective in which analysis, geometry, and dynamics are governed by stability over operational procedures, rather than by fixed abstract constructions.

Classical structures are recovered as invariant regimes corresponding to sufficiently rich classes of modes, while deviations from invariance give rise to anisotropy, degeneracy, and mode-sensitivity.

From this perspective, the mode class R determines not only the limiting functional, but also the admissible differential structure, the induced geometry, and the resulting dynamical behavior. In this sense, convergence modes act as a unifying structural parameter controlling analytical, geometric, and variational phenomena.

The present work establishes the foundational layer of this framework. Several directions remain open, including:

- a fully developed mode-dependent measure theory,
- a rigorous functional-analytic formulation of mode gradients and flows,
- a systematic connection with Γ-convergence and variational limits,
- computational and numerical implications of mode-dependent optimization.

Remark 35. *From the perspective of mode-based analysis, convergence of variational and dynamical processes is governed not solely by smoothness or convexity assumptions, but by stability with respect to admissible classes of convergence modes.*

In this sense, classical gradient flows and minimizing movements correspond to mode-invariant regimes, while deviations from invariance reflect sensitivity to discretization, regularization, and operational structure.

Acknowledgements

This work continues the development of Mode-Based Analysis initiated in Part I [1].

A Appendix: Explicit Proof of Non-Equivalence of Gradient Flows

We provide a fully explicit proof that the classical gradient flow and the mode-dependent gradient flow may produce different asymptotic behavior.

Theorem 5 (Non-equivalence of gradient flows in H^1_{per})**.** *Let $\Omega = (0,1)^2$ with periodic boundary conditions, and consider the functional*

$$J[u] = \int_\Omega |\nabla u|^2 \, dx$$

defined on $H^1_{\mathrm{per}}(\Omega)$.

Let R_x be a mode class admitting only increments in the x-direction.

Then there exists initial data $u_0 \in H^1_{\mathrm{per}}(\Omega)$ such that:

- *the classical gradient flow*
$$\partial_t u = \Delta u$$
converges to 0 in $L^2(\Omega)$ as $t \to \infty$,
- *the mode-dependent gradient flow*
$$\partial_t u = \partial_{xx} u$$
remains constant in time.

In particular, the two flows have different limiting behavior.

Proof. Let
$$u_0(x, y) = \sin(2\pi y).$$

Clearly, $u_0 \in H^1_{\mathrm{per}}(\Omega)$.

Step 1: Classical gradient flow.

The classical gradient flow is the heat equation:

$$\partial_t u = \Delta u.$$

We compute:

$$\Delta u_0 = \partial_{xx} u_0 + \partial_{yy} u_0 = 0 - (2\pi)^2 \sin(2\pi y) = -4\pi^2 \sin(2\pi y).$$

Thus u_0 is an eigenfunction of the Laplacian with eigenvalue $-4\pi^2$.
Therefore the solution is:

$$u(t, x, y) = e^{-4\pi^2 t} \sin(2\pi y).$$

Hence

$$\|u(t, \cdot, \cdot)\|_{L^2(\Omega)} = e^{-4\pi^2 t} \|u_0\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Step 2: Mode-dependent gradient flow.
For the mode class R_x, only x-variations are admissible. The corresponding gradient flow is

$$\partial_t u = \partial_{xx} u.$$

We compute:

$$\partial_{xx} u_0 = 0,$$

since u_0 is independent of x.
Thus u_0 is a stationary solution:

$$\partial_t u = 0.$$

Therefore

$$u(t, x, y) = \sin(2\pi y) \quad \text{for all } t \geq 0.$$

Step 3: Comparison.
The classical flow converges to 0, while the mode-dependent flow remains equal to the initial condition.
Thus the two flows have different asymptotic limits. □

Remark 36. *This example shows that the mode-dependent gradient flow preserves components orthogonal to the admissible tangent cone, whereas the classical flow dissipates them.*

References

[1] A. A. Nekludoff, *Mode-dependent differentials and stability of limits in partial differential equations*, 2026. DOI: 10.5281/zenodo.19842439

[2] L. C. Evans, *Partial Differential Equations.* American Mathematical Society, 2010.

[3] A. Braides, *Gamma-Convergence for Beginners.* Oxford University Press, 2002.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2000.